A characterization of recognizable picture languages by tilings by finite sets[†]

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Abstract

As extension of the Kleene star to pictures, we introduce the operation of tiling. We give a characterization of recognizable picture languages by intersection of tilings by finite sets of pictures.

1 Introduction

In this paper, we considerer picture languages as sets of rectangular arrays of symbols. Many formalisms have been introduced to generalize the notion of recognizable languages to the case of picture languages. In [6], D. Giammarresi and A. Restivo introduce the notion of local picture languages which is direct extension of local string languages. They then define recognizable picture languages by projection of local picture languages — that is a well-known property in string languages.

This class is very interesting since it corresponds to the class of picture languages recognizable by a particular class of cellular automata called on-line tessellation automata [13], and to the class of picture languages definable by existential expressions in monadic second-order logic [9]. A survey of the topic is given in the "Handbook of Formal Languages" [8].

In the string language theory, it is well known [4, 14, 15, 19] that any recognizable string language can be obtained by projection of the intersection of the star of two finite languages. Formally, if a string language R over Σ is recognizable, there exist two finite string languages A and B over an alphabet Σ' and a letter-to-letter morphism $\varphi : \Sigma'^* \to \Sigma^*$ such that:

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$$R^* = \varphi(A^* \cap B^*)$$

It turns out that for any $R \subseteq \Sigma^*$, R is recognizable if and only if there exist a letter-to-letter morphism $\varphi : \Sigma'^* \to \Sigma^*$ and two finite sets $A, B \subset (\Sigma' \cup \{\#\})^*$ such that:

$$R = \{\varphi(\omega) \mid \#\omega\# \in A^* \cap B^*\}$$

We are interested in finding a similar characterization of recognizable picture languages. To do that, we need to extend the Kleene star to pictures. We propose a extension in terms of tilings. Given a picture language L, the Kleene star of L, denoted by L^{**} of this language is the set of all pictures that can be tiled by pictures of L.

2 Preliminaries and notations

We assume the reader to be familiar with basic formal language theory (see [3, 5] for more precisions). For picture languages, we recall some definitions from [8]. Let Σ be a finite alphabet. A picture over Σ is a two-dimensional rectangular array of letters of Σ. We denote the set of all pictures over Σ by Σ^{**}.

For a picture p of size (n, m), where n is the number of rows and m the number of columns of p, we denote by $p(i, j)$ the letter of Σ which occurs in i^{th} row and j^{th} column (starting in the left-top corner). The set of all the pictures over Σ of size (n, m) is denoted by $\Sigma^{n,m}$. We denote by $\tilde{p}$, also called bounded picture of p, the $(n+2, m+2)$ picture over $\Sigma \cup \{\#\}$, where $\#$ is a special letter which does not belong to Σ, defined by:

1. $\forall 1 \leq i \leq n+2 \quad \tilde{p}(i, 1) = \tilde{p}(i, n+2) = \#$
2. $\forall 1 \leq j \leq m+2 \quad \tilde{p}(1, j) = \tilde{p}(m+2, j) = \#$
3. $\forall 2 \leq i \leq n+1, 2 \leq j \leq m+1 \quad \tilde{p}(i, j) = p(i-1, j-1)$

For instance, if we consider the alphabet $\Sigma = \{a, b, c\}$, we have for the picture p :

$$p = \begin{array}{|c|c|c|}\hline a & b & a \\ \hline b & c & c \\ \hline\end{array} \qquad \tilde{p} = \begin{array}{|c|c|c|c|c|}\hline \# & \# & \# & \# & \# \\ \hline \# & a & b & a & \# \\ \hline \# & b & c & c & \# \\ \hline \# & \# & \# & \# & \# \\ \hline\end{array}$$

Let p be a picture of size (n, m) over an alphabet Σ. For $r \leq n$ and $s \leq m$, we denote by $T_{r,s}(p)$ the set of the (r, s) sub-pictures of p:

$$T_{r,s}(p) = \left\{ q \in \Sigma^{r,s} \mid \begin{array}{c} \exists 0 \leq x \leq n-r, 0 \leq y \leq m-s \quad \forall 1 \leq i \leq r, 1 \leq j \leq s \\ q(i, j) = p(x+i, y+j) \end{array} \right\}$$

With pictures we have two concatenation products. Let p be a (n,m) picture and p' be a (n',m') picture. The row concatenation of p with p', that is denoted by $p \ominus p'$, is defined if and only if $m = m'$ and is the $(n+n',m)$ picture satisfying:

$$\begin{array}{ll} \forall 1 \leq i \leq n \; \forall 1 \leq j \leq m & (p \ominus p')(i,j) = p(i,j) \\ \forall 1 \leq i \leq n' \; \forall 1 \leq j \leq m & (p \ominus p')(n+i,j) = p'(i,j) \end{array}$$

In the same way, the column concatenation of p with p', that is denoted by $p \oplus p'$, is defined if and only if $n = n'$ and is the $(n, m+m')$ picture satisfying:

$$\begin{array}{ll} \forall 1 \leq i \leq n \; \forall 1 \leq j \leq m & (p \oplus p')(i,j) = p(i,j) \\ \forall 1 \leq i \leq n \; \forall 1 \leq j \leq m' & (p \oplus p')(i,m+j) = p'(i,j) \end{array}$$

We also define a clock-wise rotation over pictures. Let $p \in \Sigma^{m,n}$ be a picture. The rotation of p, denoted by P^R is a picture over Σ of size (n,m) such that:

$$p^R = \boxed{\begin{array}{ccc} p_{m,1} & \cdots & p_{1,1} \\ \vdots & & \vdots \\ p_{m,n} & \cdots & p_{1,n} \end{array}}$$

A picture language over Σ is a subset of Σ^{**}. The operation of concatenation and rotation are extended to languages. Let L and K be two picture languages.

$$\begin{array}{rcl} L \ominus K & = & \{p \ominus q \mid p \in L, q \in K\} \\ L \oplus K & = & \{p \oplus q \mid p \in L, q \in K\} \\ L^R & = & \{p^R \mid p \in L\} \end{array}$$

Iteration of column concatenation or row concatenation define the star operators. Let L be a picture language.

$$\begin{array}{lll} L^{\ominus 1} = L & L^{\ominus i+1} = L \ominus L^{\ominus i} & L^{\ominus *} = \bigcup_{i \geq 1} L^{\ominus i} \\ L^{\oplus 1} = L & L^{\oplus i+1} = L \oplus L^{\oplus i} & L^{\oplus *} = \bigcup_{i \geq 1} L^{\oplus i} \end{array}$$

Let L be a picture language. We define $T_{r,s}(L) = \bigcup_{p \in L} T_{r,s}(p)$. The definition of local picture languages is a direct extension of the notion of local string languages.

Definition 2.1 *Let L be a picture language over Σ. The language L is local if there exists a set Δ of $(2,2)$ pictures over $\Sigma \cup \{\#\}$ such that $L = \{p \in \Sigma^{**} \mid T_{2,2}(\tilde{p}) \subseteq \Delta\}$.*

The class of all local picture languages over an alphabet Σ is denoted by $\mathrm{Loc}(\Sigma^{**})$.

We know that every recognizable string language is the image by a letter-to-letter morphism of a local string language. We then need to define mapping in pictures. Let Σ and Σ' be two finite alphabets and let $\pi : \Sigma \to \Sigma'$ be a mapping. The projection by π of a picture $p \in \Sigma^{n,m}$ is the picture $p' \in \Sigma'^{n,m}$ such that for all $1 \leq i \leq n, 1 \leq j \leq m \quad p'(i,j) = \pi(p(i,j))$. We denote $p' = \pi(p)$. By extension, we denote by $\pi(L)$ the projection by mapping by π of the language L over Σ and $\pi(L) = \{p' \in \Sigma'^{**} \mid \exists p \in L \quad p' = \pi(p)\}$.

Definition 2.2 *Let L be a picture language over Σ. The language L is recognizable if there exist a local picture language L' over Σ' and a mapping $\pi : \Sigma' \rightarrow \Sigma$ such that $L = \pi(L')$.*

The set of all recognizable picture languages over an alphabet Σ is denoted by $\mathrm{Rec}(\Sigma^{**})$. In [16], we also give a finer characterization of recognizable picture languages by using the so-called hv-local picture languages.

Definition 2.3 *Let $L \subseteq \Sigma^{**}$ be a picture language. The language L is hv-local if there exists a set Δ of horizontal and vertical dominoes over $\Sigma \cup \{\#\}$ such that $L = \{q \in \Sigma^{**} \mid T_{1,2}(\tilde{q}) \cup T_{2,1}(\tilde{q}) \subseteq \Delta\}$.*

The set of all hv-local picture languages over an alphabet Σ is denoted by $\text{hv-Loc}(\Sigma^{**})$.

Proposition 2.4 *Let $L \subseteq \Sigma^{**}$ be a picture language. The language L is recognizable if and only if there exist a hv-local picture language L' over Σ' and a mapping $\pi : \Sigma' \rightarrow \Sigma$ such that $L = \pi(L')$.*

The main implication of this result is the possibility of treating recognizable picture languages with well-known tools of string language theory by using row-column combination (for more details, see [16]).

Definition 2.5 *Let A and B be two string languages over an alphabet Σ. The row-column combination of A and B, denoted by $A \oplus B$, is the picture language containing all pictures p such that: all rows of p (taken as string) belong to A and all columns of p (taken as string from top to bottom) belong to B.*

Proposition 2.6 *Let L be a picture language over Σ. The picture language L is recognizable if and only if there exist two recognizable string languages A and B over an alphabet X and a projection π from X into Σ such that $L = \pi(A \oplus B)$.*

The class $\mathrm{Rec}(\Sigma^{**})$ contains finite picture languages and is closed by concatenation, star operators, union, intersection and rotation.

The last tool we need is the Cartesian product of two picture languages.

Definition 2.7 *Let $p \in \Sigma^{**}$ and $q \in \Sigma'^{**}$ be two pictures. The Cartesian product of p and q, denoted by $p \otimes q$, is defined only if $\mathrm{size}(p) = \mathrm{size}(q)$ and is the picture $f \in (\Sigma \times \Sigma')^{**}$ of the same size satisfying:*

$$\forall 1 \leq i \leq \mathrm{row}(p), 1 \leq j \leq \mathrm{col}(p) \quad f(i,j) = (p(i,j), q(i,j))$$

Let $L \subseteq \Sigma^{**}$ and $K \subseteq \Sigma'^{**}$ be two picture languages. The Cartesian product of L and K is the picture language over $\Sigma \times \Sigma'$ defined by:

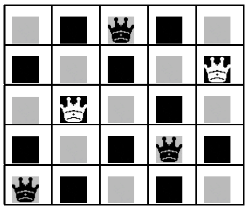
$$L \otimes K = \{f \in (\Sigma \times \Sigma')^{**} \mid \exists p \in L, q \in K \quad f = p \otimes q\}$$

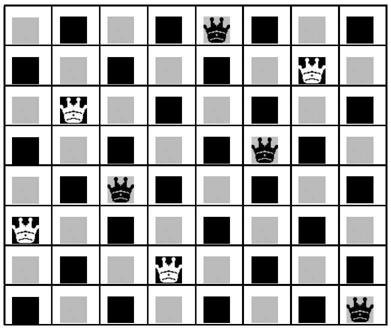
Proposition 2.8 *Let $L \in \mathrm{Rec}(X^{**})$ and $K \in \mathrm{Rec}(Y^{**})$ be two recognizable picture languages. The language $L \otimes K$ is a recognizable picture language over $X \times Y$.*

3 An example of recognizable language

Before continuing, and in order to manipulate the tools we have introduced, we are going to treat the k-queens problem by showing that the set of pictures which represent chessboards (squares) of any size where there are as many queens as rows and such that no queen strike another queen is a recognizable picture language. This example is inspired by [17] where the 8-queens problem is treated.

For instance, with the alphabet $\Sigma = \{$■, ▒, ♛, ♕$\}$, the two following pictures are in this language we denote L:

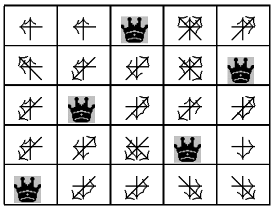




For the moment, we forget the colors of the chessboard and we study only the positions of the queens. In the associated local language, denoted by L'_1, it suffices to mark the struck squares with arrows designed by $X = \{\uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow, \leftarrow, \nwarrow\}$. For instance, the strikes of the higher queens in the 5×5 chessboard are noted:

$\leftarrow$	$\leftarrow$	♛	$\rightarrow$	$\rightarrow$
	$\swarrow$	$\downarrow$	$\searrow$	♛
$\swarrow$	♛	$\downarrow$		$\searrow$
		$\downarrow$	♛	
♛		$\downarrow$		

Since a square can be (and should be) struck by several queens, we use the subsets of X that we denote by $Y = 2^X$. For the chessboard of the previous example, we have:



It suffices, then to verify (a) that from each queen starts all arrows, (b) that each arrow comes from an arrow in the same direction or from a queen, (c) that no arrow strike a queen and (d) that each arrow points an arrow in the same direction or a border. All these verifications are clearly local. We note L_1 the image of this local by the projection which associates a white square with letters of Y and a queen with a queen (we note $\Sigma' = \{\square, \text{♛}\}$); this recognizable language contains all pictures of Σ'^{**} where no queen is checked.

Nevertheless, we have to verify that we have a square and that there is as many queens as lines. To do that, we build a language L_2 which contains all squares that have a queen by line and by column. We define $L_2 = \square^{*}\text{♛}\square^{*} \oplus \square^{*}\text{♛}\square^{*}$ and $L_3 = L_1 \cap L_2$. And yet, we just have to color the chessboard.

We use the Cartesian product of L_3 with L_4 which contains all chessboards (with no queen and of size greater than $(2,2)$) over the alphabet $\Sigma'' = \{\blacksquare, \square\}$. We define:

$$L'_4 = \left(\left(\begin{array}{|c|c|}\hline \square & \blacksquare \\ \hline \blacksquare & \square \\ \hline \end{array} \right)^{\obar *} \right)^{\ominus *}$$

In order to obtain L_4, it suffices to take:

$$L_4 = L'_4 \cup \left(L'_4 \cup L'_4 \obar \begin{array}{|c|}\hline \square \\ \hline \blacksquare \\ \hline \end{array}^{\ominus *} \right) \ominus \left(\begin{array}{|c|c|}\hline \square & \blacksquare \\ \hline \end{array}^{\obar *} \cup \begin{array}{|c|c|}\hline \square & \blacksquare \\ \hline \end{array}^{\obar *} \obar \begin{array}{|c|}\hline \square \\ \hline \end{array} \right)$$

We define $L_5 = L_3 \otimes L_4$ which is recognizable and we consider the projection π from $\Sigma' \times \Sigma''$ into Σ :

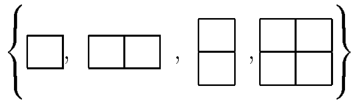
$$\begin{array}{rcl} \pi((\square, \blacksquare)) & = & \blacksquare \\ \pi((\square, \square)) & = & \square \\ \pi((\text{♛}, \blacksquare)) & = & \text{♕} \\ \pi((\text{♛}, \square)) & = & \text{♛} \end{array}$$

We have $L = \pi(L_5)$ which is a recognizable picture language.

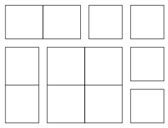
4 Tiling operations

In order to define tilings using a set of pictures, we can use concatenation and star operations we have already seen. Nevertheless, as we can see in the following example, these definitions do not correspond to the intuitive idea of tiling.

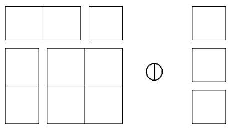
Example. *We consider the picture language L which contains pictures with the following shape:*

$L =$ 

Hence, $(L^{\ominus *})^{\obar *}$ and $(L^{\obar *})^{\ominus *}$ do not contain the picture p with the shape:

$p =$ 

In fact, p belongs to $(L^{\ominus *})^{\obar *}$ if there exist in $L^{\ominus *}$ i pictures $p_1, p_2, \ldots, p_i$ such that $p = p_1 \obar p_2 \obar \ldots \obar p_i$. But for p, there is a unique vertical factorization:

$p =$ 

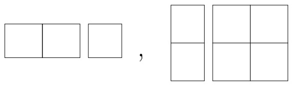
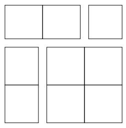
and the first picture does not belong to $L^{\ominus *}$.

We can define another star operation we denote by $^{\ominus\obar *}$.

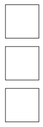
Definition 4.1 *Let $L \subseteq \Sigma^{**}$ be a picture language. We define:*

1. $L^{\ominus\obar 1} = L$,
2. $L^{\ominus\obar i+1} = L^{\ominus\obar i} \cup L^{\ominus\obar i} \ominus L^{\ominus\obar i} \cup L^{\ominus\obar i} \obar L^{\ominus\obar i}$,
3. $L^{\ominus\obar *} = \bigcup_{i \geq 1} L^{\ominus\obar i}$.

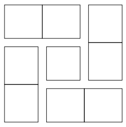
Example. *We consider the picture set L and the picture p from the previous example, we have $p \in L^{\Theta\Phi *}$, since:*

 $\in L^{\Theta\Phi 2} \Rightarrow$  $\in L^{\Theta\Phi 3}$

and

 $\in L^{\Theta\Phi 3} \Rightarrow p \in L^{\Theta\Phi 4} \subset L^{\Theta\Phi *}$

Nevertheless, the following picture p' does not belong to $L^{\Theta\Phi *}$ since it cannot be obtained by concatenation of pictures of L:

$p' =$ 

A coherent "tiling" which contains pictures like the one given in the example consists in finding a partition of the picture where each component is a picture of the first set.

Definition 4.2 *Let p be a picture of Σ^{**}. A sub-domain of p is a set with the form $\{x, \ldots, x'\} \times \{y, \ldots, y'\}$ such that $1 \leq x \leq x' \leq \mathrm{row}(p)$ and $1 \leq y \leq y' \leq \mathrm{col}(p)$. We denote by $D(p)$ the set of all sub-domain of p.*

Let $d = \{x, \ldots, x'\} \times \{y, \ldots, y'\} \in D(p)$ be a sub-domain of p. The sub-picture of p associated with d, denoted by $\mathrm{spic}(p, d)$, is the picture of size $(x' - x + 1, y' - y + 1)$ such that:

$$\forall 1 \leq i \leq x' - x + 1 \;\forall 1 \leq j \leq y' - y + 1 \quad \mathrm{spic}(p, d)(i, j) = p(x + i - 1, y + j + 1)$$

Example. *With the alphabet $\Sigma = \{a, b, c\}$, the picture:*

$p =$

b	c	a	b	b	a
a	a	b	c	a	c
a	b	a	c	a	b
c	a	c	a	c	a

and the sub-domain $d = \{2,3,4\} \times \{3,4\}$, we have:

$$\mathrm{spic}(p,d) = \begin{array}{|c|c|}\hline b & c \\ \hline a & c \\ \hline c & a \\ \hline \end{array}$$

Definition 4.3 *Let L be a picture language over Σ. The set of all tilings by L, also called L-tilings and denoted by L^{**}, is the picture language which contains all pictures p satisfying:*

$$\begin{array}{c} \exists d_1, \ldots, d_k \in D(p) \text{ a partition of } \{1, \ldots, \mathrm{row}(p)\} \times \{1, \ldots, \mathrm{col}(p)\} \\ \text{such that} \\ \forall 1 \leq i \leq k \quad \mathrm{spic}(p, d_i) \in L \end{array}$$

This definition is coherent with the usual notion of tiling used, for example, to tile polyominoes or the plane [1, 2, 10, 11, 12]. O. Matz defines other kind of "tilings" obtained by concatenation in [18].

Note that if the pictures of L have the same size, we have $(L^{\ominus *})^{\oplus *} = (L^{\oplus *})^{\ominus *} = L^{\ominus\oplus *} = L^{**}$; then the notation L^{**} is coherent with the fact that Σ^{**} denotes the set of all possible pictures over the alphabet Σ.

5 Closure of recognizable languages by tiling

D. Giammarresi and A. Restivo have shown in [7] that the set of all pictures obtained by tiling by a finite set of polyominoes is recognizable. With the same idea, it is straightforward to see that the language obtained by tiling by a finite picture language is also recognizable. In this section we extend this result to tilings by a recognizable picture languages.

Proposition 5.1 *Let $L \in \mathrm{Rec}(\Sigma^{**})$ be a recognizable picture language. The language L^{**} is recognizable.*

Proof: It is clear that it suffices to show that the L-tiling with L hv-local is recognizable. Let $L \in$ hv-Loc(Σ^{**}) a hv-local picture language. We denote by Δ the set of authorized dominoes of L.

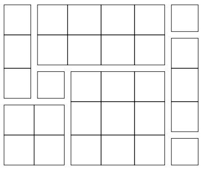
Let C_4 be the picture language over $X = \{1,2,3,4\}$ where each 4-connected component (of the same color) is a rectangle. The language C_4 is local. The associated authorized domino set is Θ defined by:

$$\Theta = \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \mid a,b,c,d \in X \cup \{\#\} \quad \forall x \in X \quad |a.b.c.d|_x \neq 3 \right\}$$

It means that tiles with exactly three identical letters are prohibited. For instance, the following tiles do not belong to Θ:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 4 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 2 & 2 \\ \hline \end{array}$$

Let p be a picture and a partition of p. Since the graph which links neighboring (considering the Von Neuman's neighboring) components of the partition is a planar graph, we can color this graph with 4 colors. It follows that for each partition, we can find a corresponding picture in C_4. At reversal, to each picture of C_4 corresponds a partition:



$$\leftrightarrow \quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & 2 & 1 \\ \hline 1 & 2 & 2 & 2 & 2 & 3 \\ \hline 1 & 3 & 4 & 4 & 4 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 & 2 \\ \hline \end{array}$$

Let K be the picture language $\Sigma^{**} \otimes C_4$. By using Proposition 2.8, we know that K is recognizable. We define a set of dominoes $\Delta' = \Delta'_h \cup \Delta'_v$ constructed from Δ:

$$\begin{aligned}
\Delta'_h \quad = \quad & \left\{ \begin{array}{|c|c|}\hline (a,i) & (b,i) \\ \hline \end{array} \mid a, b \in \Sigma, i \in X \quad \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \in \Delta \right\} \\
\cup & \left\{ \begin{array}{|c|c|}\hline \# & (a,i) \\ \hline \end{array} \mid a \in \Sigma, i \in X \quad \begin{array}{|c|c|}\hline \# & a \\ \hline \end{array} \in \Delta \right\} \\
\cup & \left\{ \begin{array}{|c|c|}\hline (a,i) & \# \\ \hline \end{array} \mid a \in \Sigma, i \in X \quad \begin{array}{|c|c|}\hline a & \# \\ \hline \end{array} \in \Delta \right\} \\
\cup & \left\{ \begin{array}{|c|c|}\hline (a,i) & (b,j) \\ \hline \end{array} \mid a, b \in \Sigma, i \neq j \in X \quad \begin{array}{|c|c|}\hline a & \# \\ \hline \end{array}, \begin{array}{|c|c|}\hline \# & b \\ \hline \end{array} \in \Delta \right\}
\end{aligned}$$

$$\begin{aligned}
\Delta'_v \quad = \quad & \left\{ \begin{array}{|c|}\hline (a,i) \\ \hline (b,i) \\ \hline \end{array} \mid a, b \in \Sigma, i \in X \quad \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \in \Delta \right\} \\
\cup & \left\{ \begin{array}{|c|}\hline \# \\ \hline (a,i) \\ \hline \end{array} \mid a \in \Sigma, i \in X \quad \begin{array}{|c|}\hline \# \\ \hline a \\ \hline \end{array} \in \Delta \right\} \\
\cup & \left\{ \begin{array}{|c|}\hline (a,i) \\ \hline \# \\ \hline \end{array} \mid a \in \Sigma, i \in X \quad \begin{array}{|c|}\hline a \\ \hline \# \\ \hline \end{array} \in \Delta \right\} \\
\cup & \left\{ \begin{array}{|c|}\hline (a,i) \\ \hline (b,j) \\ \hline \end{array} \mid a, b \in \Sigma, i \neq j \in X \quad \begin{array}{|c|}\hline a \\ \hline \# \\ \hline \end{array}, \begin{array}{|c|}\hline \# \\ \hline b \\ \hline \end{array} \in \Delta \right\}
\end{aligned}$$

Let K' the subset of K defined by:

$$K' = \{p \in K \mid T_{1,2}(\tilde{p}) \cup T_{2,1}(\tilde{p}) \subseteq \Delta'\}$$

The tiles of Δ' simply verify that two adjacent cells in the same component — colored with the same letter of X — can appear in L and that two adjacent cells which are not in the same component can be placed in the border in L. It is straightforward to see that K' is recognizable. That follows that the image of K' by the projection $\pi : \Sigma \times X \to \Sigma$ which associates a with (a, i) is also recognizable. It is clear that $L^{**} = \pi(K') \in \mathrm{Rec}(\Sigma^{**})$. □

We can notice that it is an open question to know whether or not the class Rec is closed by the operation $\ominus \mathbin{\oplus} *$.

6 Characterization of recognizable picture languages

In the string language theory, it is well known [4, 14, 15, 19] that any recognizable string language can be obtained by projection of the intersection of the star of two finite languages. Formally, if a string language R over Σ is recognizable, there exist two finite string languages A and B over an alphabet Σ' and a letter-to-letter morphism $\varphi : \Sigma'^* \to \Sigma^*$ such that:

$$R^* = \varphi(A^* \cap B^*)$$

It turns out that for any $R \in \mathrm{Rec}(\Sigma^*)$, there exist a letter-to-letter morphism $\varphi : \Sigma'^* \to \Sigma^*$ and two finite sets $A, B \subset (\Sigma' \cup \{\#\})^*$ such that:

$$R = \{\varphi(\omega) \mid \#\omega\# \in A^* \cap B^*\}$$

In this section, we try to extend this result to two-dimensional case by using the operation of tiling we have defined.

Proposition 6.1 *Let $L \subseteq \Sigma^{**}$ a picture language. The language L is recognizable if and only if there exist two (finite) sets of $(\Sigma' \cup \{\#\})^{2,2} \cup \{\boxed{\#}\}$ denoted by A_1, and A_2 and a projection $\pi : \Sigma' \to \Sigma$ such that:*

$$L = \{\pi(p) \mid \tilde{p} \in A_1^{**} \cap A_2^{**}\}$$

Proof: Since the family of recognizable picture language contains finite sets and is closed by tiling, intersection and projection, it is clear that the left-to-right implication is true. Conversely, it suffices to show that for any hv-local picture language L, we can construct the sets A_1, A_2 and the projection π which satisfy the property. We denote by Δ the set of authorized dominoes in L, and from Δ we define the set Δ' of authorized $(2,2)$ tiles in L (any hv-local language is local):

$$\Delta' = \{p \in (\Sigma \cup \{\#\})^{2,2} \mid T_{2,1}(p) \subseteq \Delta \wedge T_{1,2}(p) \subseteq \Delta\}$$

The idea of the proof is similar to the one in string languages. From the local language we construct the sets A_1 and A_2 so that for all bounded picture the tiling by A_1 begins on the $(1,1)$ cell and the tiling by A_2 on the $(2,2)$ cell. To do this, we color the cells with four distinct colors.

We define:

$$\Sigma' = \{a_1, a_2, a_3, a_4 \mid a \in \Sigma\}$$

The index corresponds to "colors" we impose to square. We want to obtain a coloring like:

#	#	#	#	#	#	#
#	$-_1$	$-_2$	$-_1$	$-_2$	$-_1$	#
#	$-_3$	$-_4$	$-_3$	$-_4$	$-_3$	#
#	$-_1$	$-_2$	$-_1$	$-_2$	$-_1$	#
#	$-_3$	$-_4$	$-_3$	$-_4$	$-_3$	#
#	#	#	#	#	#	#

We denote by π the projection from $(\Sigma' \cup \{\#\}) \rightarrow (\Sigma \cup \{\#\})$ which associates # with # and a with $a_i \in \Sigma'$. We design by Σ_i the set of all letters of Σ' whose index is i ($i \in \{1,2,3,4\}$).

Only the tilings by A_1 can begin on $(1,1)$:

$$A_1 = \pi^{-1}(\Delta') \cap \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \mid \begin{array}{ll} a \in \{\#\} \cup \Sigma_4 & b \in \{\#\} \cup \Sigma_3 \\ c \in \{\#\} \cup \Sigma_2 & d \in \{\#\} \cup \Sigma_1 \end{array} \right\} \cup \{\boxed{\#}\}$$

And the tiling by A_2 should begin on $(2,2)$:

$$\begin{array}{rcl} A_2 & = & \pi^{-1}(\Delta') \cap \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \mid \begin{array}{ll} a \in \{\#\} \cup \Sigma_1 & b \in \{\#\} \cup \Sigma_2 \\ c \in \{\#\} \cup \Sigma_3 & d \in \{\#\} \cup \Sigma_4 \end{array} \right\} \\ & & \setminus \left\{ \begin{array}{|c|c|}\hline \# & a \\ \hline \# & b \\ \hline \end{array} \mid \begin{array}{l} a \in \{\#\} \cup \Sigma_4 \\ b \in \{\#\} \cup \Sigma_2 \end{array} \right\} \\ & & \setminus \left\{ \begin{array}{|c|c|}\hline \# & \# \\ \hline a & b \\ \hline \end{array} \mid a \in \{\#\} \cup \Sigma_4 \quad b \in \{\#\} \cup \Sigma_3 \right\} \\ & & \cup \{\boxed{\#}\} \end{array}$$

Let p' be a picture of $\Sigma'^{m,n}$ such that $\tilde{p'}$ belongs to A_1^{**}. We have:

$$\forall p' \in \Sigma'^{m,n}$$

$$\begin{gathered} \tilde{p'} \in A_1^{**} \\ \Downarrow \\ \forall 2 \leq i \leq m+1 \; \forall 2 \leq j \leq n+1 \end{gathered}$$

$$\begin{cases} \tilde{p'}(i,j) \in \Sigma_1 \quad \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i-1,j-1) & \tilde{p'}(i-1,j) \\ \hline \tilde{p'}(i,j-1) & \tilde{p'}(i,j) \\ \hline \end{array} \in A_1 \\ \tilde{p'}(i,j) \in \Sigma_2 \quad \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i-1,j) & \tilde{p'}(i-1,j+1) \\ \hline \tilde{p'}(i,j) & \tilde{p'}(i,j+1) \\ \hline \end{array} \in A_1 \\ \tilde{p'}(i,j) \in \Sigma_3 \quad \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i,j-1) & \tilde{p'}(i,j) \\ \hline \tilde{p'}(i+1,j-1) & \tilde{p'}(i+1,j) \\ \hline \end{array} \in A_1 \\ \tilde{p'}(i,j) \in \Sigma_4 \quad \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i,j) & \tilde{p'}(i,j+1) \\ \hline \tilde{p'}(i+1,j) & \tilde{p'}(i+1,j+1) \\ \hline \end{array} \in A_1 \end{cases} \qquad \textbf{(P1)}$$

In the same way, if p' belongs to A_2^{**}, we have:

$$\forall p' \in \Sigma'^{m,n}$$

$$\begin{gathered} \tilde{p'} \in A_2^{**} \\ \Downarrow \\ \forall 2 \leq i \leq m+1 \; \forall 2 \leq j \leq n+1 \end{gathered}$$

$$\begin{cases} \tilde{p'}(i,j) \in \Sigma_4 \quad \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i-1,j-1) & \tilde{p'}(i-1,j) \\ \hline \tilde{p'}(i,j-1) & \tilde{p'}(i,j) \\ \hline \end{array} \in A_2 \\ \tilde{p'}(i,j) \in \Sigma_3 \quad \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i-1,j) & \tilde{p'}(i-1,j+1) \\ \hline \tilde{p'}(i,j) & \tilde{p'}(i,j+1) \\ \hline \end{array} \in A_2 \\ \tilde{p'}(i,j) \in \Sigma_2 \quad \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i,j-1) & \tilde{p'}(i,j) \\ \hline \tilde{p'}(i+1,j-1) & \tilde{p'}(i+1,j) \\ \hline \end{array} \in A_2 \\ \tilde{p'}(i,j) \in \Sigma_1 \quad \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i,j) & \tilde{p'}(i,j+1) \\ \hline \tilde{p'}(i+1,j) & \tilde{p'}(i+1,j+1) \\ \hline \end{array} \in A_2 \end{cases} \qquad \textbf{(P2)}$$

Let $a \in \Sigma'$ be the top left letter of p': $a = \tilde{p'}(2,2)$. This letter can belong to Σ_1 or Σ_2. Nevertheless by using definitions of the A_i and the properties (**Pi**) ($i \in \{1,2\}$), we show that the only possible case is $a \in \Sigma_1$. In fact, if $a \in \Sigma_i$ with $i \in \{1,2\}$, allowing to (**Pi**), we have:

$$\begin{array}{|c|c|}\hline \# & \# \\ \hline \# & a \\ \hline \end{array} \in A_i$$

Since such tiles only occur in A_1, a belongs to Σ_1.

With the same reasoning, we show that if $\tilde{p'}$ belongs to A_i^{**} for $i \in \{1,2\}$, we have:

$$\forall 1 \le 2i+1 \le m \; \forall 1 \le 2j+1 \le n \quad \begin{array}{|c|c|}\hline \tilde{p'}(2i+1,2j+1) & \tilde{p'}(2i+1,2j+2) \\ \hline \tilde{p'}(2i+2,2j+1) & \tilde{p'}(2i+2,2j+2) \\ \hline \end{array} \in A_1$$

$$\forall 1 \le 2i \le m \; \forall 1 \le 2j \le n \quad \begin{array}{|c|c|}\hline \tilde{p'}(2i,2j) & \tilde{p'}(2i,2j+1) \\ \hline \tilde{p'}(2i+1,2j) & \tilde{p'}(2i+1,2j+1) \\ \hline \end{array} \in A_2 \qquad (\mathbf{P})$$

Example. *For the picture p' whose bounded picture belongs to the respective tilings by A_1 and A_2:*

$$p' = \begin{array}{|c|c|c|c|c|}\hline a_1 & b_2 & b_1 & a_2 & c_1 \\ \hline a_3 & c_4 & a_3 & b_4 & c_3 \\ \hline b_1 & b_2 & c_1 & c_2 & c_1 \\ \hline c_3 & a_4 & b_3 & c_4 & b_3 \\ \hline \end{array}$$

This different tilings are:

$$\left.\begin{array}{cccc}
\begin{array}{|c|c|}\hline \# & \# \\ \hline \# & a_1 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline b_2 & b_1 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline a_2 & c_1 \\ \hline \end{array} &
\begin{array}{|c|}\hline \# \\ \hline \# \\ \hline \end{array} \\[2ex]
\begin{array}{|c|c|}\hline \# & a_3 \\ \hline \# & b_1 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline c_4 & a_3 \\ \hline b_2 & c_1 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline b_4 & c_3 \\ \hline c_2 & c_1 \\ \hline \end{array} &
\begin{array}{|c|}\hline \# \\ \hline \# \\ \hline \end{array} \\[2ex]
\begin{array}{|c|c|}\hline \# & c_3 \\ \hline \# & \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline a_4 & b_3 \\ \hline \# & \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline c_4 & b_3 \\ \hline \# & \# \\ \hline \end{array} &
\begin{array}{|c|}\hline \# \\ \hline \# \\ \hline \end{array}
\end{array}\right. \in A_1^{**}$$

and

$$\left.\begin{array}{cccc}
\begin{array}{|c|}\hline \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline \end{array} \\[2ex]
\begin{array}{|c|}\hline \# \\ \hline \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline a_1 & b_2 \\ \hline a_3 & c_4 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline b_1 & a_2 \\ \hline a_3 & b_4 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline c_1 & \# \\ \hline c_3 & \# \\ \hline \end{array} \\[2ex]
\begin{array}{|c|}\hline \# \\ \hline \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline b_1 & b_2 \\ \hline c_3 & a_4 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline c_1 & c_2 \\ \hline b_3 & c_4 \\ \hline \end{array} &
\begin{array}{|c|c|}\hline c_1 & \# \\ \hline b_3 & \# \\ \hline \end{array} \\[2ex]
\begin{array}{|c|}\hline \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline \end{array} &
\begin{array}{|c|c|}\hline \# & \# \\ \hline \end{array}
\end{array}\right. \in A_2^{**}$$

In other words, the property $(\mathbf{P})$ means that all sub-pictures in $T_{2,2}(\tilde{p'})$ are in a A_i. The picture $p = \pi(p') \in \Sigma^{**}$ is so a picture of K since we have:

$$T_{1,2}(\tilde{p}) \subseteq \pi(T_{1,2}(A_1)) \subseteq T_{1,2}(\Delta') \subseteq \Delta$$
$$T_{2,1}(\tilde{p}) \subseteq \pi(T_{2,1}(A_1)) \subseteq T_{2,1}(\Delta') \subseteq \Delta$$

It turns out that p is a picture of L.

Conversely, for an arbitrary picture $p \in L$, it is easy to construct a picture $p' \in \pi^{-1}(p)$ such that $\tilde{p'}$ belongs to A_1^{**} and A_2^{**} — it suffices to choose the good "colors". □

By using the definition of recognizable picture languages by row-column combination (see Definition 2.5 and Proposition 2.6) by recognizable string languages, we obtain directly the following proposition.

Proposition 6.2 *Let $L \subseteq \Sigma^{**}$ be a picture language. The language L is a recognizable picture language if and if there exist two (finite) sets of pictures of $(\Sigma' \cup \{\#\})^{1,2} \cup \{\boxed{\#}\}$ denoted by A_1 and A_2, two (finite) sets of pictures of $(\Sigma' \cup \{\#\})^{2,1} \cup \{\boxed{\#}\}$ denoted by A_3 and A_4 and a projection $\pi : \Sigma' \to \Sigma$ such that:*

$$L = \{\pi(p) \mid \tilde{p} \in A_1^{**} \cap A_2^{**} \cap A_3^{**} \cap A_4^{**}\}$$

7 Strict tilings

In this section, we are interested in characterization of recognizable picture languages by the intersection of tilings by one-dimensional finite picture languages. We need to introduce some technical definitions and properties to obtain this results.

A k-uple $(A_1, \ldots, A_k)$ of picture languages over Σ is called a k-tiling. The picture language defined by this k-tiling is denoted by $\mathcal{L}(A_1, \ldots, A_k)$ and is defined by:

$$\mathcal{L}(A_1, \ldots, A_k) = \{p \in \Sigma^{**} \mid \tilde{p} \in A_1^{**} \cap \ldots \cap A_k^{**}\}$$

Definition 7.1 *Let $(A_1, \ldots, A_k)$ be a k-tiling over an alphabet Σ. This k-tiling is called strict if*

$$\mathcal{L}(A_1 \cup \boxed{a}, \ldots, A_k \cup \boxed{a}) = \mathcal{L}(A_1, \ldots, A_k) \cup \boxed{a}^{**}$$

where a is a new letter which does not belong to Σ.

It means that in the k-tiling, letters of Σ cannot be mixed with other symbols.

The following assertion clearly holds.

Assertion 7.2 *Let $(A_1, \ldots, A_k)$ be a strict k-tiling. The k-tiling $(A_1^R, \ldots, A_k^R)$ is strict and:*

$$\mathcal{L}(A_1^R, \ldots, A_k^R) = \mathcal{L}(A_1, \ldots, A_k)^R$$

Proposition 7.3 *Let L, K be two picture languages of Σ^{**} such that $L = \pi(\mathcal{L}(A_1, \ldots, A_k))$ and $K = \sigma(\mathcal{L}(B_1, \ldots, B_k))$ where the k-uples $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ are two strict k-tilings. Then, there exists a strict k-tiling $(C_1, \ldots, C_k)$ such that $L \cup K = \psi(\mathcal{L}(C_1, \ldots, C_k))$.*

Proof: We give the construction of $(C_1, \ldots, C_k)$ and ψ. We can suppose that the alphabets Σ_1 and Σ_2, of $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$, respectively, are disjoint. We take $C_i = A_1 \cup B_i$.

It is clear that if $p \in \mathcal{L}(C_1, \ldots, C_k)$, then $p \in \Sigma_1^{**}$ or $p \in \Sigma_2^{**}$. In fact, let p contains some letters of Σ_1 and Σ_2. Let p' be the image of p by the projection which replace the letters of Σ_2 by a letter a which does not belong to Σ_1. We have:

$$p' \in \mathcal{L}(A_1 \cup \boxed{a}, \ldots, A_k \cup \boxed{a})$$

There is a contradiction if p contains letters of both Σ_1 and Σ_2.

It suffices to define ψ by $\psi(a) = \pi(a)$ if $a \in \Sigma_1$ and $\psi(a) = \sigma(a)$ if $a \in \Sigma_2$. $\square$

It is easy to deduce the following weaker result.

Proposition 7.4 *Let L, K be two picture languages of Σ^{**} such that $L = \pi(\mathcal{L}(A_1, \ldots, A_k))$ and $K = \sigma(\mathcal{L}(B_1, \ldots, B_k))$ where $(A_1, \ldots, A_k)$ is a k-tiling and $(B_1, \ldots, B_k)$ a strict k-tilings. Then, there exists a k-tiling $(C_1, \ldots, C_k)$ such that $L \cup K = \psi(\mathcal{L}(C_1, \ldots, C_k))$.*

The first characterization with one-dimensional sets uses three sets of dominoes.

Lemma 7.5 *Let $L \in$ hv-Loc(Σ^{**}) be a hv-local picture language containing only pictures of height greater than one. There exists a strict 3-tiling with three (finite) sets of pictures of $(\Sigma' \cup \{\#\})^{1,2} \cup (\Sigma' \cup \{\#\})^{2,1} \cup \{\boxed{\#}\}$ denoted by A_1, A_2 and A_3, and a projection $\pi : \Sigma' \to \Sigma$ such that:*

$$L = \pi(\mathcal{L}(A_1, A_2, A_3))$$

Proof: We denote by Δ the authorized domino set associated with L. First, we define an alphabet Γ and a projection φ from $\Gamma \cup \{\#\}$ into $\Sigma \cup \{\#\}$:

$$\begin{aligned} \Gamma &= \left\{p \in (\Sigma \cup \{\#\})^{3,3} \mid T_{2,1} \cup T_{1,2} \subseteq \Delta\right\} \\ &\quad \setminus \{\#^{\Phi 3} \ominus \Sigma^{1,3} \ominus \#^{\Phi 3}\} \end{aligned}$$

and for all $a \in \Gamma$ we define $\varphi(a) = a(2,2)$ and $\varphi(\#) = \#$. We remove from Γ the picture of size $(3,3)$ with three $\#$ on the top row and on the bottom row in order to prevent pictures with only one row.

We define the domino set Θ over Γ:

$$\Theta = \left\{ \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \mid a, b \in \Gamma \quad \forall 1 \leq i \leq 2, 1 \leq j \leq 3 \quad a(i+1, j) = b(i, j) \right\}$$

$$\cup \left\{ \begin{array}{|c|}\hline \# \\ \hline a \\ \hline\end{array} \mid a \in \Gamma \quad \forall 1 \leq i \leq 3 \quad a(1,i) = \# \right\}$$

$$\cup \left\{ \begin{array}{|c|}\hline a \\ \hline \# \\ \hline\end{array} \mid a \in \Gamma \quad \forall 1 \leq i \leq 3 \quad a(3,i) = \# \right\}$$

$$\cup \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \mid a, b \in \Gamma \quad \forall 1 \leq i \leq 3, 1 \leq j \leq 2 \quad a(i,j+1) = b(i,j) \right\}$$

$$\cup \left\{ \begin{array}{|c|c|}\hline \# & a \\ \hline\end{array} \mid a \in \Gamma \quad \forall 1 \leq i \leq 3 \quad a(i,1) = \# \right\}$$

$$\cup \left\{ \begin{array}{|c|c|}\hline a & \# \\ \hline\end{array} \mid a \in \Gamma \quad \forall 1 \leq i \leq 3 \quad a(i,3) = \# \right\}$$

It is easy to see that:

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \in \Theta \quad \Rightarrow \quad \begin{array}{|c|}\hline \varphi(a) \\ \hline \varphi(b) \\ \hline\end{array} \in \Delta$$
$$\begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \in \Theta \quad \Rightarrow \quad \begin{array}{|c|c|}\hline \varphi(a) & \varphi(b) \\ \hline\end{array} \in \Delta \tag{1}$$

As in the proof of the Proposition 6.1, we are going to color the picture in order to impose tilings to have a particular disposition. The tilings by A_1 will be composed of vertical dominoes and will start on the first row; the tilings by A_2 will be also composed of vertical dominoes but will start on the second row; the tilings by A_3, which will contain horizontal dominoes, will start on the first column for even row and on second column for odd row.

We use four colors and we define the alphabet $\Gamma' = \{a_i \mid a \in \Gamma, i \in \{1,2,3,4\}\}$. We want to obtain a colouring like the following one:

#	#	#	#	#	#	#
#	$-_1$	$-_2$	$-_1$	$-_2$	$-_1$	#
#	$-_3$	$-_4$	$-_3$	$-_4$	$-_3$	#
#	$-_1$	$-_2$	$-_1$	$-_2$	$-_1$	#
#	$-_3$	$-_4$	$-_3$	$-_4$	$-_3$	#
#	#	#	#	#	#	#

We denote by ψ the projection from $(\Gamma' \cup \{\#\}) \to (\Gamma \cup \{\#\})$ which associates # with # and a with $a_i \in \Gamma'$. We design by Γ_i the set of all letters of Γ' whose index is i ($i \in \{1,2,3,4\}$).

The tree tilings are defined by:

$$A_1 = \left\{ \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \in \psi^{-1}(\Theta) \mid a \in \Gamma_3 \cup \{\#\}, b \in \Gamma_1 \cup \{\#\} \right\}$$

$$\cup \left\{ \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \in \psi^{-1}(\Theta) \mid a \in \Gamma_4 \cup \{\#\}, b \in \Gamma_2 \cup \{\#\} \right\} \cup \{\boxed{\#}\}$$

$$\begin{aligned} A_2 &= \left\{ \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \in \psi^{-1}(\Theta) \mid a \in \Gamma_1 \cup \{\#\}, b \in \Gamma_3 \cup \{\#\} \right\} \\ &\cup \left\{ \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \in \psi^{-1}(\Theta) \mid a \in \Gamma_2 \cup \{\#\}, b \in \Gamma_4 \cup \{\#\} \right\} \cup \{\boxed{\#}\} \\ &\setminus \left\{ \begin{array}{|c|}\hline \# \\ \hline a \\ \hline \end{array} \mid a \in \Gamma_3 \cup \Gamma_4 \right\} \end{aligned}$$

$$\begin{aligned} A_3 &= \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \in \psi^{-1}(\Theta) \mid a \in \Gamma_2 \cup \{\#\}, b \in \Gamma_1 \cup \{\#\} \right\} \\ &\cup \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \in \psi^{-1}(\Theta) \mid a \in \Gamma_3 \cup \{\#\}, b \in \Gamma_4 \cup \{\#\} \right\} \cup \{\boxed{\#}\} \\ &\setminus \left\{ \begin{array}{|c|c|}\hline \# & a \\ \hline \end{array} \mid a \in \Gamma_4 \right\} \end{aligned}$$

Then, we have:

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \in A_1 \cup A_2 \Rightarrow \begin{array}{|c|}\hline \psi(a) \\ \hline \psi(b) \\ \hline \end{array} \in \Theta \qquad \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \in A_3 \Rightarrow \begin{array}{|c|c|}\hline \psi(a) & \psi(b) \\ \hline \end{array} \in \Theta \tag{2}$$

Let π be the mapping from $\Gamma' \cup \{\#\}$ into $\Sigma \cup \{\#\}$ defined by $\pi = \varphi \circ \psi$. By using the properties (1) and (2), we obtain:

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array} \in A_1 \cup A_2 \Rightarrow \begin{array}{|c|}\hline \pi(a) \\ \hline \pi(b) \\ \hline \end{array} \in \Delta \qquad \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \in A_3 \Rightarrow \begin{array}{|c|c|}\hline \pi(a) & \pi(b) \\ \hline \end{array} \in \Delta \tag{3}$$

With this definitions of A_1, A_2, A_3 and π, we show that: 1. A_1, A_2 and A_3 define a 3-tiling satisfying the desired colouring; 2. for a picture $p' \in \Gamma'^{**}$, if $\tilde{p'} \in A_1^{**} \cap A_2^{**} \cap A_3^{**}$ then $p = \pi(p')$ belongs to L; 3. for a picture $p \in L$, there exists a picture $p' \in \pi^{-1}(p)$ such that $\tilde{p'} \in A_1^{**} \cap A_2^{**} \cap A_3^{**}$; (4) the 3-tiling is strict.

We use the three following properties which are deduced from the definitions of A_1, A_2 and A_3.

$$\forall p' \in {\Gamma'}^{m,n}$$

$$\tilde{p'} \in A_1^{**}$$
$$\Downarrow$$
$$\forall 2 \leq i \leq m+1 \; \forall 2 \leq j \leq n+1$$

$$\left\{ \begin{array}{lcl} \tilde{p'}(i,j) \in \Gamma_1 & \Rightarrow & \begin{array}{|c|}\hline \tilde{p'}(i-1,j) \\ \hline \tilde{p'}(i,j) \\ \hline \end{array} \in A_1 \\ \tilde{p'}(i,j) \in \Gamma_2 & \Rightarrow & \begin{array}{|c|}\hline \tilde{p'}(i-1,j) \\ \hline \tilde{p'}(i,j) \\ \hline \end{array} \in A_1 \\ \tilde{p'}(i,j) \in \Gamma_3 & \Rightarrow & \begin{array}{|c|}\hline \tilde{p'}(i,j) \\ \hline \tilde{p'}(i+1,j) \\ \hline \end{array} \in A_1 \\ \tilde{p'}(i,j) \in \Gamma_4 & \Rightarrow & \begin{array}{|c|}\hline \tilde{p'}(i,j) \\ \hline \tilde{p'}(i+1,j) \\ \hline \end{array} \in A_1 \end{array} \right. \qquad (\mathbf{P1})$$

$$\forall p' \in {\Gamma'}^{m,n}$$

$$\tilde{p'} \in A_2^{**}$$
$$\Downarrow$$
$$\forall 2 \leq i \leq m+1 \; \forall 2 \leq j \leq n+1$$

$$\left\{ \begin{array}{lcl} \tilde{p'}(i,j) \in \Gamma_1 & \Rightarrow & \begin{array}{|c|}\hline \tilde{p'}(i,j) \\ \hline \tilde{p'}(i+1,j) \\ \hline \end{array} \in A_2 \\ \tilde{p'}(i,j) \in \Gamma_2 & \Rightarrow & \begin{array}{|c|}\hline \tilde{p'}(i,j) \\ \hline \tilde{p'}(i+1,j) \\ \hline \end{array} \in A_2 \\ \tilde{p'}(i,j) \in \Gamma_3 & \Rightarrow & \begin{array}{|c|}\hline \tilde{p'}(i,j) \\ \hline \tilde{p'}(i,j) \\ \hline \end{array} \in A_2 \\ \tilde{p'}(i-1,j) \in \Gamma_4 & \Rightarrow & \begin{array}{|c|}\hline \tilde{p'}(i-1,j) \\ \hline \tilde{p'}(i,j) \\ \hline \end{array} \in A_2 \end{array} \right. \qquad (\mathbf{P2})$$

$$\forall p' \in {\Gamma'}^{m,n}$$

$$\tilde{p'} \in A_3^{**}$$
$$\Downarrow$$
$$\forall 2 \leq i \leq m+1 \; \forall 2 \leq j \leq n+1$$

$$\left\{ \begin{array}{lcl} \tilde{p'}(i,j) \in \Gamma_1 & \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i,j-1) & \tilde{p'}(i,j) \\ \hline \end{array} \in A_3 \\ \tilde{p'}(i,j) \in \Gamma_2 & \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i,j) & \tilde{p'}(i,j+1) \\ \hline \end{array} \in A_3 \\ \tilde{p'}(i,j) \in \Gamma_3 & \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i,j) & \tilde{p'}(i,j+1) \\ \hline \end{array} \in A_3 \\ \tilde{p'}(i,j) \in \Gamma_4 & \Rightarrow & \begin{array}{|c|c|}\hline \tilde{p'}(i,j-1) & \tilde{p'}(i,j) \\ \hline \end{array} \in A_3 \end{array} \right. \qquad (\mathbf{P3})$$

1. (A_1, A_2, A_3) is a 3-tiling satisfying the desired colouring.

   Let $p' \in \mathcal{L}(A_1, A_2, A_3)$ be a picture in the 3-tiling. Let a' the first letter of p', $a' = p'(1,1)$.

   For the moment, we show that the height of p' is greater than one. We suppose that its height is one and we show that it leads to a contradiction. By using the properties (**P1**) and (**P2**), we can deduce that:

   $$\begin{array}{|c|}\hline \# \\ \hline a' \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline a' \\ \hline \# \\ \hline\end{array}\ \in A_1 \cup A_2$$

   Then, according to the property (2), we have:

   $$\begin{array}{|c|}\hline \# \\ \hline \psi(a') \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline \psi(a') \\ \hline \# \\ \hline\end{array}\ \in \Theta$$

   The letter $a = \psi(a')$ belongs to Γ and by definition of Θ, for all $i \in \{1,2,3\}$, we have $a(1,i) = a(3,i) = \#$. There is a contradiction since this letter does not belong to Γ.

   Now that we know that the height of p' is greater than one, we are interested in the colouring. We show that the letter a' belongs to Γ_1. In fact, a' cannot be colored by 2, 3 and 4:

   (a) $a' \in \Gamma_2$: we denote by b' the letter under a'. According to (**P2**), we have:

   $$\begin{array}{|c|}\hline a' \\ \hline b' \\ \hline\end{array}\ \in A_2$$

   We know that b' belongs to Γ' since the height of p' is not one, then by definition of A_2, b' belongs to Γ_4 and by (**P3**), we have:

   $$\begin{array}{|c|c|}\hline \# & b' \\ \hline\end{array}\ \in A_3$$

   It is not possible since this dominoes do not appear in A_3 (see definition).

   (b) $a' \in \Gamma_3 \cup \Gamma_4$: by (**P2**), we have a contradiction:

   $$\begin{array}{|c|}\hline \# \\ \hline a' \\ \hline\end{array}\ \in A_2$$

   Hence, we know that $a' \in A_1$. By using the properties (**P1**), (**P2**) and (**P3**), it is easy to show that the rest of the colouring is correct.

2. for each picture $p' \in \mathcal{L}(A_1, A_2, A_3)$ then $p = \pi(p')$ belongs to L.

   We denote by (m, n) the size of p'. It suffices to show the two following properties:

$$\forall 1 \leq i \leq m+1,\ 2 \leq j \leq n+1 \quad \begin{array}{|c|}\hline \tilde{p}(i,j) \\ \hline \tilde{p}(i+1,j) \\ \hline\end{array} \in \Delta \qquad (\mathbf{V})$$

$$\forall 2 \leq i \leq m+1,\ 1 \leq j \leq n+1 \quad \begin{array}{|c|c|}\hline \tilde{p}(i,j) & \tilde{p}(i+1,j) \\ \hline\end{array} \in \Delta \qquad (\mathbf{H})$$

   (a) Let $1 \leq i \leq m+1, 2 \leq j \leq n+1$ be a position in $\tilde{p'}$, we show the property $(\mathbf{V})$. We denote by a' and b' the letters $\tilde{p'}(i,j)$ and $\tilde{p'}(i+1,j)$, respectively. Three cases can happen:

      i. $a' = \#$. It means that $i = 1$. Since the colouring is correct, we know that $b' \in \Gamma_1 \cup \Gamma_2$ and by using $(\mathbf{P1})$, we have:

$$\begin{array}{|c|}\hline a' \\ \hline b' \\ \hline\end{array} \in A_1$$

      ii. $a' \in \Gamma_1 \cup \Gamma_2$. We use the property $(\mathbf{P2})$ and deduce that:

$$\begin{array}{|c|}\hline a' \\ \hline b' \\ \hline\end{array} \in A_2$$

      iii. $a' \in \Gamma_3 \cup \Gamma_4$. We use the property $(\mathbf{P1})$ and deduce that:

$$\begin{array}{|c|}\hline a' \\ \hline b' \\ \hline\end{array} \in A_1$$

   In all cases, the vertical domino is in A_1 or A_2. According to (3), the property $(\mathbf{V})$ holds.

   (b) Let $2 \leq i \leq m+1, 1 \leq j \leq n+1$ be a position in $\tilde{p'}$, we show the property $(\mathbf{H})$. We denote by a' and b' the letters $\tilde{p'}(i,j)$ and $\tilde{p'}(i,j+1)$, respectively. The projection of a' and b' into Γ by ψ (which remove the color) are denoted by a and b respectively. Several cases can happen:

      i. $a' \in \Gamma_2 \cup \Gamma_3$ or $b' \in \Gamma_1 \cup \Gamma_2$. According to $(\mathbf{P3})$ and (3), we have:

$$\begin{array}{|c|c|}\hline a' & b' \\ \hline\end{array} \in A_2 \Rightarrow \begin{array}{|c|c|}\hline \pi(a') & \pi(b') \\ \hline\end{array} = \begin{array}{|c|c|}\hline \tilde{p}(i,j) & \tilde{p}(i+1,j) \\ \hline\end{array} \in \Delta$$

      ii. $a' \in \Gamma_1$ and $i \neq m+1$. By using $(\mathbf{P2})$ and (2), we deduce that:

$$\begin{array}{|c|}\hline a' \\ \hline \tilde{p'}(i+1,j) \\ \hline\end{array} \in A_2 \text{ and } \tilde{p'}(i+1,j) \in \Gamma_3$$

$$\begin{array}{|c|}\hline a \\ \hline \psi(\tilde{p'}(i+1,j)) \\ \hline\end{array} \in \Theta$$

According to (**P3**), we have:

$$\begin{array}{|c|c|}\hline \tilde{p'}(i+1,j) & \tilde{p'}(i+1,j+1) \\ \hline\end{array} \in A_3 \text{ and } \tilde{p'}(i+1,j) \in \Gamma_4 \cup \{\#\}$$

$$\begin{array}{|c|c|}\hline \psi(\tilde{p'}(i+1,j)) & \psi(\tilde{p'}(i+1,j+1)) \\ \hline\end{array} \in \Theta$$

We denote by c' and d' the letters of Γ' $\tilde{p'}(i+1,j)$ and $\tilde{p'}(i+1,j+1)$. Their image by ψ are respectively denoted by c and d. There are two sub-cases:

A. $d' = \#$. It means that $j = n+1$ and we have $b' = \#$. By definition of Θ, we have:

$$\begin{array}{rr} \forall k \in \{2,3\}, l \in \{1,2,3\} & a(k,l) = c(k-1,l) \\ \forall k \in \{1,2,3\} & c(k,3) = \# \end{array}$$

It turns out that the letter c have the following shape:

$$c = \begin{array}{|c|c|c|}\hline & a(2,2) & \# \\ \hline & c(2,2) & \# \\ \hline & & \# \\ \hline\end{array}$$

By definition of Γ, the dominoes which occur in c belong to Δ. We then have:

$$\begin{array}{|c|c|}\hline a(2,2) & \# \\ \hline\end{array} = \begin{array}{|c|c|}\hline \pi(a) & \# \\ \hline\end{array} = \begin{array}{|c|c|}\hline \tilde{p}(i,j) & \tilde{p}(i+1,j) \\ \hline\end{array} \in \Delta$$

B. $d' \in \Gamma_4$. According to (**P2**) and (2), we have:

$$\begin{array}{|c|}\hline b' \\ \hline d' \\ \hline\end{array} \in A_2 \Rightarrow \begin{array}{|c|}\hline b \\ \hline d \\ \hline\end{array} \in \Theta$$

By definition of Θ, we deduce:

$$c = \begin{array}{|c|c|c|}\hline & a(2,2) & b(2,2) \\ \hline & c(2,2) & d(2,2) \\ \hline & & \\ \hline\end{array} \quad d = \begin{array}{|c|c|c|}\hline a(2,2) & b(2,2) & \\ \hline c(2,2) & d(2,2) & \\ \hline & & \\ \hline\end{array}$$

And, by definition of Γ, we have:

$$\begin{array}{|c|c|}\hline a(2,2) & b(2,2) \\ \hline\end{array} = \begin{array}{|c|c|}\hline \pi(a) & \pi(b) \\ \hline\end{array} = \begin{array}{|c|c|}\hline \tilde{p}(i,j) & \tilde{p}(i+1,j) \\ \hline\end{array} \in \Delta$$

iii. $a' \in \Gamma_4$ and $i \neq m+1$. Similar to previous case.

iv. $a' \in \Gamma_1 \cup \Gamma_4$ and $i = m+1$. Similar to previous cases but we take c' and d' be equals to $\tilde{p'}(i-1,j)$ and $\tilde{p'}(i-1,j+1)$ respectively.

v. $a' = \#$ and $b' \in \Gamma_3$. Similar to the case 2(b)iiA (where $d' = \#$).

In all cases we reach to prove that the horizontal domino belongs to $\mathbf{\Delta}$, then the property $(\mathbf{H})$ holds.

It shows that

$$\{\pi(p) \mid \tilde{p} \in A_1^{**} \cap A_2^{**} \cap A_3^{**} \cap A_4^{**}\} \subseteq L$$

3. for a picture $p \in L$, there exists a picture $p' \in \pi^{-1}(p)$ such that $\tilde{p'} \in A_1^{**} \cap A_2^{**} \cap A_3^{**}$.

   We define a mapping σ from Σ^{**} into Γ^{**}:

$$\begin{array}{rccl} \sigma: & \Sigma^{**} & \longrightarrow & \Gamma^{**} \\ & p & \longmapsto & q \in \Gamma^{**} \text{ with } \mathrm{size}(p') = \mathrm{size}(p) \text{ and:} \end{array}$$

$$\forall 1 \leq i \leq \mathrm{col}(p) \quad \forall 1 \leq j \leq \mathrm{row}(p)$$

$$q(i,j) = \begin{array}{|c|c|c|} \hline \tilde{p}(i,j) & \tilde{p}(i,j+1) & \tilde{p}(i,j+2) \\ \hline \tilde{p}(i+1,j) & \tilde{p}(i+1,j+1) & \tilde{p}(i+1,j+2) \\ \hline \tilde{p}(i+2,j) & \tilde{p}(i+2,j+1) & \tilde{p}(i+2,j+2) \\ \hline \end{array}$$

   Let p be a picture of L. It suffices to take a good colouring q' of $q = \sigma(p)$ to have

$$\tilde{q'} \in A_1^{**} \cap A_2^{**} \cap A_3^{**}$$

   It shows that

$$L \subseteq \{\pi(p) \mid \tilde{p} \in A_1^{**} \cap A_2^{**} \cap A_3^{**} \cap A_4^{**}\}$$

4. the 3-tiling is strict.

   If we add to A_1, A_2 and A_3 a picture $\boxed{a}$ with $a \neq \Gamma'$, the properties $(\mathbf{P1})$, $(\mathbf{P2})$ and $(\mathbf{P3})$ always hold.

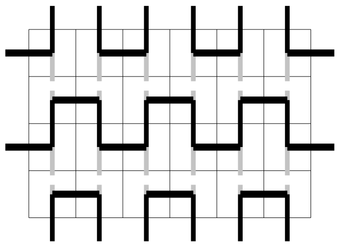
   Let $p' \in (\Gamma' \cup \{a\})^{**}$ be a picture such that

$$\tilde{p'} \in A_1^{**} \cap A_2^{**} \cap A_3^{**}$$

   If p' contains a letters of Γ', it is easy to see, by using the properties $(\mathbf{P1})$, $(\mathbf{P2})$ and $(\mathbf{P3})$ that p' belongs to Γ'^{**}. Then, (A_1, A_2, A_3) is a strict 3-tiling.

$\square$

We give comments of the proof in order to simplify next proofs. For a picture in the 3-tiling, it corresponds a tiling by A_1, A_2 and A_3. We can associate with this tiling a graph:



Two cells are connected with a vertical full edge if they are covered by a domino of A_1, with a vertical grey doted edge if they are covered by a domino of A_2 and with an horizontal full edge if they are covered by a domino of A_3.

This connections in the graph correspond to vertical and horizontal controls needed in hv-local languages. For missing controls (like between $(1,1)$ and $(1,2)$ cells of the previous example), we need to find a bounded path. For instance, in the previous example, we can always find a path of length smaller than 3 ($(1,1)-(2,1)-(2,2)-(1,2)$ to link $(1,1)$ and $(1,2)$). The fact that there always exists a such path allows us to transmit the needed information puted in letters of the alphabet.

If for all tilings there is some edges which cross the borders of the picture, then the k-tiling is strict.

Theorem 7.6 *Let $L \subseteq \Sigma^{**}$ a picture language. The language L is recognizable if and only if there exist three (finite) sets of $(\Sigma' \cup \{\#\})^{2,2} \cup \{\boxed{\ \#\ }\}$ denoted by A_1, A_2 and A_3 and a projection $\pi : \Sigma' \to \Sigma$ such that:*

$$L = \{\pi(p) \mid \tilde{p} \in A_1^{**} \cap A_2^{**} \cap A_3^{**}\}$$

Proof: Let L_1, L_2 and L_3 be the three picture languages defined by:

$$\begin{aligned} L_1 &= \{p \in L \mid \mathrm{row}(p) > 1\} \\ L_2 &= \{p \in L \mid \mathrm{col}(p) > 1\} \\ L_3 &= \{p \in L \mid \mathrm{row}(p) = \mathrm{col}(p) = 1\} \end{aligned}$$

It is easy to see that L_1, L_2 and L_3 are recognizable and that $L = L_1 \cup L_2 \cup L_3$.

According to Lemma 7.5, there exist two strict 3-tilings (B_1, B_2, B_3) and (C_1, C_2, C_3) and two projection π_1 and π_2 such that

$$\begin{array}{rcl} L_1 & = & \pi_1(\mathcal{L}(B_1, B_2, B_3)) \\ L_2^R & = & \pi_2(\mathcal{L}(C_1, C_2, C_3)) \end{array}$$

By using Assertion 7.2 and Proposition 7.3, we can deduce that there exist a strict 3-tiling (D_1, D_2, D_3) and a projection φ such that

$$L_1 \cup L_2 = \varphi(\mathcal{L}(D_1, D_2, D_3))$$

It remains to show that there exist a 3-tiling (E_1, E_2, E_3) which corresponds to L_3. It suffices to take the sets defined by:

$$E_1 = \{ \boxed{\begin{array}{c} \# \\ \hline a \end{array}} \mid \boxed{a} \in L_3\} \cup \boxed{\#}$$

$$E_2 = \{ \boxed{\begin{array}{c|c} \# & a \end{array}} \mid \boxed{a} \in L_3\} \cup \boxed{\#}$$

$$E_3 = \{ \boxed{\begin{array}{c|c} a & \# \end{array}} \mid \boxed{a} \in L_3\} \cup \boxed{\#}$$

By using Proposition 7.4 the theorem holds. □

In the previous theorem we use three sets of dominoes. Now, we are looking to a characterization with two sets of triminoes.

Lemma 7.7 *Let $L \in \text{hv-Loc}(\Sigma^{**})$ be a hv-local picture language containing such that each picture p in L satisfies the following condition:*

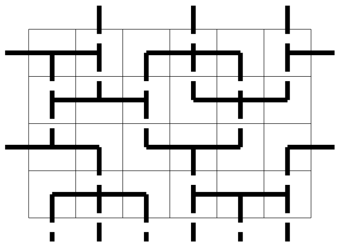
$$\begin{array}{c} \text{row}(p) > 1 \quad \text{col}(p) > 1 \\ \text{size}(p) \notin \left\{ \begin{array}{c} (3+6n, 6+6m), (4+6m, 7+6m), (6+6n, 4+6m), \\ (7+6n, 3+6m), (7+6m, 5+6m) \end{array} \mid n, m \in \mathbb{N} \right\} \end{array} \quad \textbf{(C)}$$

There exist a strict 2-tiling with two (finite) sets of pictures of $(\Sigma' \cup \{\#\})^{1,3} \cup (\Sigma' \cup \{\#\})^{1,2} \cup (\Sigma' \cup \{\#\})^{3,1} \cup (\Sigma' \cup \{\#\})^{2,1} \cup \{\boxed{\#}\}$ denoted by A_1 and A_2, and a projection $\pi : \Sigma' \to \Sigma$ such that:

$$L = \pi(\mathcal{L}(A_1, A_2))$$

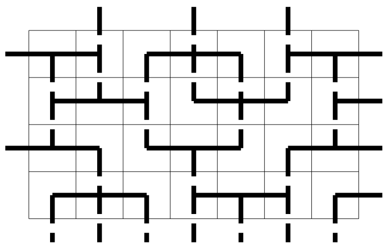
Proof: We just give the idea of the proof which is very similar to the proof of Lemma 7.5. The tiling by A_1, which contains vertical triminoes and dominoes will begin on the second row for even column and on first row for odd column. The tiling by A_2, which contains horizontal triminoes and dominoes will begin on the first column for even column and on second column for odd column.

Hence the graph associated to a tiling is like this:



This graph is always connected for pictures satisfying the condition (**C**) and there exists always a path of length smaller than 9 which links two adjoining cells.

For pictures which do not satisfy the condition, the graph is not connected. For instance, a picture of size $(4,7)$, we have:

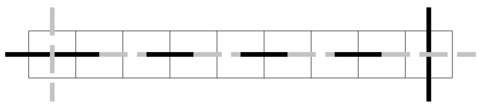


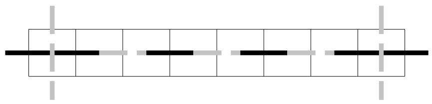
The cell in $(4,7)$ cannot be linked with the cell $(3,7)$. □

Lemma 7.8 *Let* $L \in$ hv-Loc(Σ^{**}) *be a hv-local picture language containing only pictures of height one. There exist a strict* 2*-tiling with two (finite) sets of pictures of* $(\Sigma' \cup \{\#\})^{1,3} \cup (\Sigma' \cup \{\#\})^{1,2} \cup (\Sigma' \cup \{\#\})^{3,1} \cup (\Sigma' \cup \{\#\})^{2,1} \cup \{\boxed{\#}\}$ *denoted by* A_1 *and* A_2*, and a projection* $\pi : \Sigma' \to \Sigma$ *such that:*

$$L = \pi(\mathcal{L}(A_1, A_2))$$

Proof: We give the construction with two examples. The tiling depends of the parity of the width:





Full lines represent tiles of A_1 and doted grey lines represent tiles of A_2. □

Theorem 7.9 *Let $L \subseteq \Sigma^{**}$ a picture language. The language L is recognizable if and only if there exist two (finite) sets of $(\Sigma' \cup \{\#\})^{1,3} \cup (\Sigma' \cup \{\#\})^{1,2} \cup (\Sigma' \cup \{\#\})^{3,1} \cup (\Sigma' \cup \{\#\})^{2,1} \cup \{\boxed{\#}\}$ denoted by A_1 and A_2 and a projection $\pi : \Sigma' \to \Sigma$ such that:*

$$L = \{\pi(p) \mid \tilde{p} \in A_1^{**} \cap A_2^{**}\}$$

Proof: Let L_1, L_2 and L_3 be the three picture languages defined by:

$$\begin{aligned} L_1 &= \{p \in L \mid p \text{ satisfies } (\mathbf{C})\} \\ L_2 &= \{p \in L \mid p^R \text{ satisfies } (\mathbf{C})\} \\ L_3 &= \{p \in L \mid \mathrm{row}(p) = 1\} \\ L_4 &= \{p \in L \mid \mathrm{col}(p) = 1\} \end{aligned}$$

It is easy to see that this languages are recognizable and that $L = L_1 \cup L_2 \cup L_3 \cup L_4$. Lemma 7.7 allows us to construction two strict 2-tilings corresponding to L_1 and L_2^R respectively. And by using Lemma 7.8 we can construct two string 2-tilings corresponding to L_3 and L_4^R respectively.

It suffices to apply Assertion 7.2 and Proposition 7.3 to obtain the result.
□

8 Conclusion

We succeed in characterizing recognizable picture languages with three sets of dominoes or with two sets of triminoes. Despite to the fact that it is not shown, it seems to be difficult to obtain the same result with two sets of dominoes. Nevertheless, we have some hope to obtain a characterization using a set of triminoes and a set of dominoes.

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